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HBT correlations and charge ratios in multiple production of pions

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Abstract. The influence of the HTB effect on the multiplicity distribution and charge ratios of independently produced pions is studied. It is shown that, for a wide class of models, there is a critical point, where the average number of pions becomes very large and the multiplicity distribution becomes very broad. In this regime unusual charge ratios ("centauros", "anticentauros") are strongly enhanced. The prospects for reaching this regime are discussed.

1 Introduction

It is now well established that the HBT correlations [1] influence significantly the momentum distribution of particles created in high-energy collisions. The effect on multiplicity distributions and on particle ratios, however, although predicted theoretically [2–5], has not yet been found in accelerator experiments [6]. In view of the increasing interest in measurements of multiplicity distributions in future collider experiments, we found it worthwhile to reexamine this problem once more.

We begin by reviewing the basic ideas of the HBT effect, as applied to processes of particle production. This will permit to explain our assumptions and to introduce the notation.[1]

Let $\psi_0(q_1, q_2, \ldots, q_n, \alpha) \equiv \psi_0(q, \alpha)$ be the probability amplitude for the production of n particles with momenta $[q_1, q_2, \ldots, q_n] \equiv q$ *calculated ignoring the identity of particles*. Here α denotes a collection of all other quantum numbers which may be relevant to the process in question (e.g., the momenta of other particles which we do not wish to consider explicity in a "semi-inclusive" measurement). The density matrix

$$\rho_0(q, q') = \int d\alpha \psi_0(q, \alpha) \psi_0^*(q', \alpha) \tag{1}$$

gives all the availabe information about the system in question. In particular, the momentum spectrum of particles is

$$\Omega_0(q) = \int d\alpha |\psi_0(q, \alpha)|^2 = \rho_0(q, q) \; . \tag{2}$$

In the following we shall assume that $\Omega_0(q)$ is normalized to 1:

$$\int dq \Omega_0(q) = 1 \; . \tag{3}$$

Suppose now that the particles are identical. In this case the states with different permutation of particle momenta are non-distinguishable and the wave function is a sum over all permutations

$$\psi(q, \alpha) = \sum_P \psi_0(q_P, \alpha) \; , \tag{4}$$

where q_P is the set of momenta $[q_1, q_2, \ldots, q_n]$ ordered according to the permutation P of $[1, 2, \ldots, n]$. Using (1)–(4) we obtain for the distribution of momenta of identical particles

$$\Omega(q) = \frac{1}{n!} \int d\alpha |\psi(q, \alpha)|^2 = \frac{1}{n!} \sum_{P,P'} \rho_0(q_P, q_{P'}) \; . \tag{5}$$

The factor $\frac{1}{n!}$ takes care of the fact that the phase-space for n identical particles is $n!$ times smaller than the phase-space of the non-identical ones.

Equation (5) summarizes the effect of the identity of particles on the observed spectra. It is seen that to evaluate this effect it is not enough to know the spectrum $\Omega_0(q)$. The full density matrix $\rho_0(q, q')$ is necessary. Inverting this statement we observe that the measurements of the spectra of identical particles provide information on the density matrix $\rho_0(q, q')$, inaccessible otherwise. This is, in fact, the reason why these measurements are so attractive.

There are three points worth observing in (5):

(i) When momenta of all particles are equal, we obtain the simple result

$$\Omega(q_1 = q_2 = \ldots = q_n) = n! \Omega_0(q_1 = q_2 = \ldots = q_n) \; . \tag{6}$$

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[1] We will follow the approach of [7]

(ii) The normalization of $\Omega(q)$ is different from that of $\Omega_0(q)$. Generally we have

$$\int dq\Omega(q) \geq \int dq\Omega_0(q) \tag{7}$$

Since the relation between $\Omega(q)$ and $\Omega_0(q)$ depends on n, the HBT effect affects not only the shape of momentum spectrum but also the multiplicity distribution. This aspect of the problem is the subject of the present investigation.

(iii) If particles are emitted in a pure state, i.e., if $\rho_0(q,q') = \psi_0(q)\psi_0^*(q')$ and if $\psi_0(q)$ is symmetric with respect to interchange of any pair of momenta, we have

$$\Omega(q) = n!\Omega_0(q) \tag{8}$$

a really dramatic result[2]

2 HBT phenomenon in uncorrelated emission

Consider now a system of n particles emitted independently. If we ignore the identity of particles, independent emission implies that the density matrix factorizes

$$\rho_0(q,q') = \prod_{i=1}^{n} \rho_0(q_i, q_i') \ . \tag{9}$$

Introducing this into (5) we have

$$\Omega(q) = \frac{1}{n!} \sum_{P,P'} \prod_{i=1}^{n} \rho_0((q_P)_i, (q_{P'})_i) \ . \tag{10}$$

Here we are interested in the integral

$$W_n = \int dq\Omega(q) = \sum_{P} \int \prod_{i=1}^{n} (d^3q_i \rho_0(q_i, (q_P)_i)) \ . \tag{11}$$

To calculate W_n we observe that, for each permutation P the integral on the right hand side of (11) factorizes into a product of contributions from all the cycles of P (as is well-known, each permutation can be decomposed into cycles). Let us denote the contribution from a cycle of length k by C_k. We have

$$C_k = \int d^3q_1 \dots d^3q_k \rho_0(q_1,q_2)\rho_0(q_2,q_3)\dots \\ \dots \rho_0(q_{k-1},q_k)\rho_0(q_k,q_1) \tag{12}$$

It follows from (3) that $C_1 = 1$. For $k > 1$, C_k depends on the form of $\rho_0(q,q')$ and cannot be calculated without further assumptions. One can prove, however, that all C_k are positive. Indeed, one sees from (12) that

$$C_k = Tr[\rho_0]^k \ . \tag{13}$$

Since $\rho_0(q,q')$, being a density matrix, has only non-negative eigenvalues and trace one, $C_k > 0$.

The rest of the calculation if just combinatorics.

We observe first that any two permutations which have identical partitions into cycles give equal contributions. Denoting by n_k the number of occurrences of a cycle of length k in the set of permutations considered, the contribution from all of them can be written as

$$W_n' = \prod_{k=1}^{n} (C_k)^{n_k} \frac{n!}{(k!)^{n_k}} [(k-1)!]^{n_k} \frac{1}{n_k!} \\ = n! \prod_{k=1}^{n} \frac{\left(\frac{C_k}{k}\right)^{n_k}}{n_k!} \ . \tag{14}$$

In the first equality the first factor is the integral, the second is the number of partitions of the n particles among the cycles, the third is the number of ways a cycle can be constructed from k particles and the last one corrects for the permutations of whole cycles.

W_n is obtained by summing W_n' only over permutations which have partitions into cycles different from each other. This still leaves a large number of terms but – for large n – this number is much smaller than the original $n!$.

Until now we have considered a fixed multiplicity. If the multiplicity distribution calculated with identity of particles ignored is given by $P_0(n)$, the correct multiplicity distribution of identical particles is

$$P(n) = \frac{P_0(n)W_n}{\sum_m P_0(m)W_m} \tag{15}$$

For independent emission $P_0(n)$ is given by the Poisson distribution

$$P_0(n) = e^{-\nu} \frac{\nu^n}{n!} \tag{16}$$

and we obtain an elegant formula for the generating function of the multiplicity distribution:

$$\Phi(z) \equiv \sum_n P(n)z^n = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k (z^k - 1) C_k}{k} \right) . \tag{17}$$

3 General discussion of multiplicity distributions obtained from independent emission

In this section we discuss the general properties of the multiplicity distributions obtained from (17).

First, we observe that using well-known properties of the generating function we obtain from (17) for the average multiplicity

$$\langle n \rangle = \sum_{k=1}^{\infty} \nu^k C_k \ , \tag{18}$$

[2] We emphasize again that $\Omega_0(q)$ is the spectrum calculated with *identity of particles being ignored*. Some authors [8], while discussing the HBT effect for pure states, include the identity of particles already in calculation of $\Omega_0(q)$. In this case one obtains of course $\Omega(q) = \Omega_0(q)$

and for the correlation coefficients (cumulants)

$$K_p = \sum_{k=p}^{\infty} \frac{(k-1)!}{(k-p)!} C_k \nu^k \; . \tag{19}$$

All the cumulants are positive, because all the C_k are. This means in particular that the distribution is always broader than the Poisson one.

Specific properties of the distributions defined by (17) depend, of course, on the value of ν and of the cycle integrals C_k.

The first important example we would like to consider is when particles are emitted in a pure state, i.e.

$$\rho_0(q,q') = \psi(q)\psi^*(q') \; . \tag{20}$$

It follows from (12) that $C_k = 1$ for all k and the generating function becomes

$$\Phi(z) = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k(z^k-1)}{k}\right) = \frac{1-\nu}{1-\nu z} \tag{21}$$

corresponding to the geometric distribution

$$P(n) = (1-\nu)\nu^n \; , \quad \langle n \rangle = \frac{\nu}{1-\nu} \tag{22}$$

which, at the critical point $\nu \to 1$, exhibits the phenomenon of Einstein condensation. One sees from this example that the resulting multiplicity distribution has little to do with the original Poisson one, and the observed average multiplicity may dramatically differ from the initial ν. As we shall see, this is a general phenomenon.

Further discussion depends on the assumed shape of the single-particle density matrix.

The evaluation of C_k is greatly simplified if one works in the basis where the density matrix $\rho_0(q,q')$ is diagonal.

Let us first discuss the case of a discrete eigenvalue spectrum. We have

$$C_k = \sum_m \lambda_m^k \; , \quad C_1 = \sum_m \lambda_m = 1 \tag{23}$$

and thus the generating function of the multiplicity distribution can be represented as a product

$$\Phi(z) = \prod_m \Phi_m(z) \tag{24}$$

where

$$\Phi_m(z) = \frac{1-\lambda_m \nu}{1-z\lambda_m \nu} \tag{25}$$

are the generating functions of the geometric distribution (if the eigenvalue λ_m is degenerate, the corresponding factor in (24) appears g_m times, where g_m is the degeneration factor).

Thus we obtain the average multiplicity

$$\langle n \rangle = \sum_m \frac{\lambda_m \nu}{1-\lambda_m \nu} \geq \frac{\lambda_0 \nu}{1-\lambda_0 \nu} \tag{26}$$

and the cumulants

$$K_p = (p-1)! \sum_m \left(\frac{\lambda_m \nu}{1-\lambda_m \nu}\right)^p \geq (p-1)! \left(\frac{\lambda_0 \nu}{1-\lambda_0 \nu}\right)^p \tag{27}$$

where λ_0 is the largest eigenvalue of $\rho_0(q,q')$. One sees that $\nu\lambda_0 = 1$ is the critical point of the multiplicity distribution (cf. [2]).

Let us now discuss the case when the eigenvalue spectrum is continuous. Denoting by λ the eigenvalues of $\rho_0(q,q')$ and by $\sigma(\lambda)$ the spectral functions we have

$$C_k = \int_0^1 \sigma(\lambda)\lambda^k d\lambda \; . \tag{28}$$

The normalization condition (3) implies that

$$C_1 = \int_0^1 \sigma(\lambda)\lambda d\lambda = 1 \; . \tag{29}$$

One sees that the problem reduces to a discussion of a single non-negative function $\sigma(\lambda)$ defined in the interval $[0,1]$ and satisfying the condition (29). We also note that the eigenvalues equal to zero do not contribute to C_k. One can thus always add to $\sigma(\lambda)$ a term of the form $a\delta(\lambda)$ (with an arbitrary positive constant a) without changing the results.

To be more specific, we consider the generic spectral function in the form

$$\sigma(\lambda) = \lambda_0^{-2} \frac{\Gamma(a+b+3)}{\Gamma(a+2)\Gamma(b+1)} \left(\frac{\lambda}{\lambda_0}\right)^a \left(1-\frac{\lambda}{\lambda_0}\right)^b \tag{30}$$

for $0 \leq \lambda \leq \lambda_0$ and zero otherwise.

We obtain

$$C_k = \lambda_0^{k-1} \frac{\Gamma(a+b+3)\Gamma(a+k+1)}{\Gamma(a+2)\Gamma(a+b+k+2)} \tag{31}$$

and the cumulants

$$K_p = \frac{\Gamma(a+b+3)\Gamma(a+p+1)\Gamma(p)}{\Gamma(a+b+p+2)\Gamma(a+2)} \nu^p \lambda_0^{p-1} F(p, a+p+1; a+b+p+2; \lambda_0\nu) \; . \tag{32}$$

The hypergeometric function F becomes singular at $\nu\lambda_0 = 1$ when p exceeds $b+1$.

Three special cases are of interest:

(i) $b \to -1$. In this case $\sigma(\lambda) \to \frac{1}{\lambda_0}\delta(\lambda-\lambda_0)$ and we have $C_k = \lambda_0^{k-1}$. Thus

$$\Phi(z) = \left(\frac{1-\lambda_0\nu}{1-\lambda_0\nu z}\right)^{\frac{1}{\lambda_0}} \tag{33}$$

and we recognize the negative binomial distribution with the average $\langle n \rangle = \frac{\nu}{1-\lambda_0\nu}$. For $\lambda_0 \to 0$ we recover the Poisson distribution (16). As discussed in the previous section, $\lambda_0 \to 1$ corresponds to the pure state $C_k = 1$.

(ii) $a = b = 0$. Now we have $C_k = \frac{2}{k+1}\lambda_0^{k-1}$ and one finds the following formula for the cumulants of the distribution

$$K_p = 2\frac{(p-1)!}{p+1}\nu^p \left(\frac{\lambda_0}{1-\lambda_0\nu}\right)^{p-1} F(1,2;p+2;\lambda_0\nu) \ . \quad (34)$$

From the known properties of the hypergeometric function [9] we deduce that at the critical point $\lambda_0\nu \to 1$, $\langle n \rangle$ diverges logarithmically, whereas all other cumulants diverge as negative powers of $(1-\lambda_0\nu)$.

(iii) $b > 0$. In this case the average multiplicity approaches a finite limit at the critical point $\lambda_0\nu \to 1$. The divergences appear in cumulants of order $p > b+1$.

To summarize, the distribution becomes always singular when $\nu\lambda_0$ (i.e. the product of the initial average multiplicity and the maximal eigenvaule of the density matrix) approaches 1. The character of the singularity at this critical point, however, depends crucially on the behaviour of the spectral function in the vicinity of λ_0.

4 Gaussian density matrix

In this section we consider the single particle density matrix of the Gaussian form, discussed already in several papers by Pratt [2–4] (see also [5])

$$\rho_0(q,q') = \rho_x(q_x,q'_x)\rho_y(q_y,q'_y)\rho_z(q_z,q'_z) \quad (35)$$

with

$$\rho_x(q_x,q'_x) = \left(\frac{1}{2\pi\Delta_x^2}\right)^{\frac{1}{2}} e^{-\frac{(q_x^+)^2}{2\Delta_x^2} - \frac{1}{2}R_x^2(q_x^-)^2}, \quad (36)$$

where

$$q^+ \equiv \frac{1}{2}(q+q') \ ; \quad q^- \equiv q - q' \ . \quad (37)$$

Analogous formulae define ρ_y and ρ_z. As easily seen, Δ_x^2 is the average value of the square of the x-component of the particle momentum, and R_x^2 is the average value of the square of the x-coordinate of the particle emission point. The uncertainty principle implies that for $i = x,y,z$,

$$R_i\Delta_i \geq \frac{1}{2} \quad (38)$$

In order to determine the multiplicity distribution, we first find the eigenvalues of the density matrix (36). To this end we observe that the eigenfunctions of (36) are of the form[3]

$$f_m(q) = e^{-\frac{1}{2}\frac{R}{\Delta}q^2} H_m\left(\sqrt{\frac{R}{\Delta}}q\right) \ , \quad (39)$$

where $H_m(q)$ is the Hermite polynomial of order m. Using (39) it is not difficult to find the eigenvalues:

$$\lambda_m = \lambda_0(1-\lambda_0)^m \ , \quad m = 0,1,\ldots, \quad (40)$$

where

$$\lambda_0 = \frac{2}{(1+2\Delta_x R_x)}\frac{2}{(1+2\Delta_y R_y)}\frac{2}{(1+2\Delta_z R_z)} \quad (41)$$

is the greatest of the eigenvalues. Note that (38) implies that $\lambda_0 \leq 1$, as necessary.

The generating function is given by (24), (25) and the cumulants by (27) with λ_m given by (40).

Using (40) and following the arguments of the previous section we also obtain the elegant formula

$$C_k = \sum_m \lambda_m^k = \frac{\lambda_0^k}{1-(1-\lambda_0)^k} \ . \quad (42)$$

Together with (19), this gives for the cumulants

$$K_p = \sum_{k=p}^{\infty} \frac{(k-1)!}{(k-p)!}\frac{(\nu\lambda_0)^k}{1-(1-\lambda_0)^k} \ , \quad (43)$$

which diverges, as expected, at $\nu\lambda_0 \to 1$.

5 Charge ratios

It was first pointed out by Pratt [2–4] that studies of the charge ratios may be an effective way to uncover the effects of HBT correlations in multiparticle systems. This issue can be readily treated using the methods developed in Sects. 3 and 4.

We would like to discuss independent production of positive, negative and neutral pions. The main difficulty in formulating the problem is how to implement the constraint of charge conservation, as this clearly depends on the dynamics of the production process (a thorough discussion can be found in [4]). Here we restrict ourselves to two cases which, we believe, illustrate well the main point: the charge ratios obtained may drastically differ from those expected from the "uncorrected" distributions.

In the first case the constraint of charge conservation is ignored altogether. This may be justified if the system of particles we consider is a small part of a very large system. For the generating function of multiplicity distribution we thus obtain simply

$$\Phi(z_+,z_-,z_0) = \Phi(z_+)\Phi(z_-)\Phi(z_0) \ , \quad (44)$$

where $\Phi(z)$ is the generating function of the multiplicity distribution of one of the species. Introducing $n_c = n_+ + n_-$ we have

$$\Phi(z_c,z_0) = \Phi^2(z_c)\Phi(z_0) \ . \quad (45)$$

From this equation one can obtain the full joint distribution of charged and neutral pions by the usual methods. For illustration we just quote the results for two extreme cases $n_0 = 0$ ("centauros") and $n_c = 0$ ("anticentauros"):

$$P(n_0 = 0) = \Phi(0) \ , \quad P(n_c = 0) = [\Phi(0)]^2 \ . \quad (46)$$

Using now the results of the previous section we have

$$P(n_0 = 0) = \prod_m (1-\lambda_m\nu) \ , \qquad P(n_c = 0) = [P(n_0 = 0)]^2 \ . \quad (47)$$

[3] This was pointed out to us by A. Staruszkiewicz

One immediate consequence is that the production of "centauros" must be larger than that of "anticentauros". For the Gaussian density matrix we obtain, according to (40),

$$P(n_0 = 0) = \prod_m (1 - \lambda_0 \nu (1 - \lambda_0)^m) \ . \quad (48)$$

The second case we consider is when the charged particles are produced in pairs, i.e., the distribution is of the form

$$P(n_c, n_0) = \frac{[P(n_c/2)]^2 P(n_0)}{\sum_m [P(m)]^2} \ . \quad (49)$$

Thus for production of "centauros" we obtain the same formula as before. The generating function reads

$$\Phi(z_c, z_0) = \Psi(z_c)\Phi(z_0) \quad (50)$$

where

$$\Psi(z_c) = \frac{\sum_{n_c} [P(n_c/2)]^2 z_c^{n_c}}{\sum_m [P(m)]^2} \ . \quad (51)$$

The probability of creating an "anticentauro" is

$$\Psi(z_c = 0) = \frac{[\Phi(0)]^2}{\sum_m [P(m)]^2} > [\Phi(0)]^2 \quad (52)$$

and thus the production of "anticentauros" is enhanced as compared to the previous case.

These results are illustrated in Fig. 1, where the probability of the occurrence of a "centauro", $P(n_0 = 0) = \Phi(0)$, given by (48), is plotted versus the average multiplicity of the system considered for different values of the paramter $R\Delta$, which determines the maximal eigenvalue λ_0 through the relation (41). One sees that for $R\Delta \leq 1$ this probability remains substantial even for rather large values of the total multiplicity. With increasing $R\Delta$, however, it drops pretty fast even at moderate values of the average multiplicity.

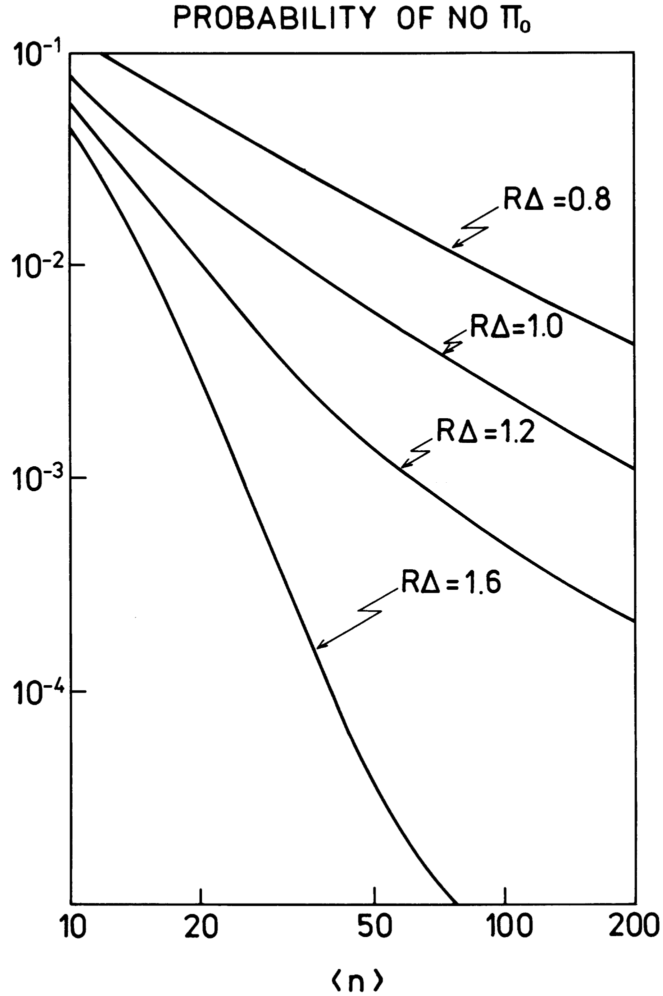

Fig. 1. Frequency of "centauro" events as function of the average total multiplicity for various values of the parameter $R\Delta$

6 Discussion and outlook

One sees from the results of the previous sections that the effects of quantum interference can modify substantially the multiplicity distribution expected naively from an uncorrelated emission of "distinguishable" particles. The modifications become spectacular when the system of particles approaches the critical point: the multiplicity distribution becomes very broad and does not resemble in any way the original Poisson distribution characteristic of uncorrelated emission. This means in particular that one expects relatively large probabilities for unusual configurations such as "centauro" or "anticentauro" events.

It is thus interesting to discuss the physical conditions for these phenomena to occur. Considering the example of the Gaussian density matrix, it is seen from Fig. 1 that the behaviour of the system is mainly determined by the parameter $R\Delta$ and that spectacular effects in charge ratios occur when $R\Delta$ is of order 1 or smaller (as already noted, c.f. (38), $R\Delta \geq \frac{1}{2}$). The condition

$$R\Delta \sim 1 \quad (53)$$

implies, for large multiplicity, either a very large particle and energy density (when R is small and Δ takes its "canonical" value of $\sim 1\,\mathrm{fm}^{-1}$), or a "canonical" energy density of about $1\,\mathrm{GeV/fm}^3$ and a very small average momentum of the particles in the c.m. of the system. To be more specific, a system of 100 pions satisfying (53) at $\Delta = 200\,\mathrm{MeV}$ would correspond to the energy density of about $35\,\mathrm{GeV}/4\,\mathrm{fm}^3 \approx 10\,\mathrm{GeV/fm}^3$. On the other hand, for an energy density of about $1\,\mathrm{GeV/fm}^3$ (and thus R correspondingly larger), Δ should not exceed 100 MeV.

Clearly, the probability of creating a "centauro" is enhanced if both effects cooperate. We thus conclude that (a) the probability of creating a "centauro" is enhanced in an environment of high energy density and that the pions emerging from the "centauros" ar likely to exhibit abnormally small relative momenta.

Finally, let us stress the crucial role played in this analysis by the "original" Poisson multiplicity distribution $P_0(n)$ and by the reference density matrix $\rho_0(q, q')$ for the distinguishable particles. This distribution is easily identified when working with Feynman diagrams[4], but it is undefined experimentally. In order to extract the HBT effect from experiment, one would like to compare the data with a reference distribution where the HBT effect has been switched off. The problem, how to define operationally this reference distribution has been discussed for many years without a generally accepted conclusion [10,11], for a review c.f. [8]. Consequently, it is not possible to check separately the assumptions about $\rho_0(q, q')$, $P_0(n)$ and the analysis of the HBT effect. In the present paper we have chosen simple, but rather general, assumptions about ρ_0, P_0 and concentrated on the HBT effect. No doubt, however, more work remains to be done in order to find a realistic "distribution of distinguishable particles" $\rho_0(q, q')$, $P_0(n)$.

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